Have a volunteer roll the black dice. Record results as: "Name", then below, "Roll 1, Roll 2".

Then, I roll the dice. If volunteer rolls different numbers, I roll two 1's. If volunteer rolls doubles, I roll normally (hope for not doubles). Record similarly.

Which set of rolls is less likely? ***They are equal, if order matters. If not, then doubles are less likely.***

Suppose (the volunteer) rolled one die, and then the other separately. What is the probability that they rolled a _3_? What about the probability that they rolled a _6_? ***P(3)=1/6, P(6)=1/6***

Did the second roll depend on the first roll? ***No, it didn't.***

Since the two events (first roll, second roll) have nothing to do with each other, we call them **independent events**.

So what is the probability that they rolled a _3_ and then a _6_? Let's make a table of every possible outcome when rolling the two dice.

	First						
Second		1	2	3	4	5	6
	1	(1,1)					
	2						
	3						
	4						(6,3)
	5						
	6			(3,6)			

How many different outcomes are there when we roll these two dice in order? ***6*6=36***

And how many ways are there to get a _3_ on the first roll, and then a _6_ on the second roll? ***Just one, (3,6)***

So what would you say is the probability of (volunteer) rolling a _3_ and then a _6_? ***1/36***

Let's see… we think the probability of getting a _3_ and then a _6_ is 1/36… the probability of rolling a _3_ first is 1/6, the probability of rolling a _6_ second is 1/6. What do you think we need to do to get the joint probability of these two independent events happening together? ***Multiply: (1/6)*(1/6) = 1/36.***

Yes! So to get the joint probability of two or more events that are independent of each other, we need to multiply the probabilities together.

So let's do the same with my rolls. What is the probability of rolling a 1? ***1/6.*** And the probability I roll a 1 the second time? ***1/6.*** So what is the probability of rolling a 1 and then another 1? ***(1/6)*(1/6) = 1/36. Also, fill in (1,1) on chart to emphasize this visually.***

So wait a minute: the probability of getting a _3_ and then a _6_ is the same as getting a _1_ and then a _1_? They're both 1/36? How does that happen? Why does it seem less likely (or more rare) to roll doubles?

It all depends on how you ask the question. Until now, we've assumed that the rolls were separate.

Let's think about rolling the dice together now. How many ways can you roll two dice and get a _3_ and a _6_? ***Refer back to table: now represented as green text. There are two ways to do this: roll a 3 and a 6 (3,6), or roll a 6 and a 3 (6,3). These are different.***

So what we're saying is that the probability of rolling a _3_ and a _6_ together is 2/36? This also simplifies to 1/18.

What about my rolls? How many ways can you roll two dice and get two ones? ***Just one: (1,1).***

So what is the probability of rolling two 1's together? ***1/36.***

Ah! So _3_ and _6_ is 2/36, and two 1's is 1/36. So it's twice as likely to roll a _3_ and a _6_ as it is to roll two 1's. Does this make sense?

So, without writing anything down, what is the relationship between P(5,4) and P(2,2)? ***You are twice as likely to roll a 5 and a 4 than you are to roll two 2's.***

Can we make any generalizations about the types of probabilities we've been talking about? ***Yes – you are twice as likely to roll any particular combination of two different numbers than you are to roll a particular number twice.***

How is this different from saying you're twice as likely to roll "not doubles" as you are to roll "doubles"? ***This isn't true; the probability of rolling doubles is P(1,1) + P(2,2) +…+ P(6,6) = 1/6 + 1/6 + … + 1/6 = 6/36 = 1/6, while the probability of rolling "not doubles" is 1-P(doubles) = 1-(1/6) = 5/6. You are five times as likely to roll "not doubles" as you are to roll "doubles".***

OK so we've learned quite a bit today. You can see that we talked about this problem in two different ways: first, we separated the rolls so that the order of the rolls made a difference in how we considered the outcomes. Second, we combined the rolls together. Making this distinction made a difference in the probabilities of the rolls we made.

In the first scenario, when we separated the rolls into a "first" roll and a "second" roll, we are talking about the **permutations** of the outcomes. Now the root of the word "permutations" is "permute". Does anyone know what it means to "permute" things? ***It means to put things in order.***

That's right – permutations have to do with outcomes where ORDER MATTERS. In our first scenario, (3,6) was not the same thing as (6,3). The _3_ had to come first, since it was rolled first. Mathematically, we would say that (3,6) and (6,3) represent two different permutations of two dice rolls.

So here's a question: how many permutations are there, in total, when you roll these two dice? ***36.***

How'd you get that answer? ***Looking at the table, there are 6 possibilities for the first roll, and 6 for the second. That makes a square of cells that is 6x6. 6*6=36.***

Alright, very good. Now, in the second scenario, we combined the rolls together. The order of the dice no longer mattered. Any guesses what we call the outcomes here? We *combined* the rolls together. ***Combinations?*** Yes!

So **combinations** represent cases where the order of events does not matter. The events are combined together. Here, we would say that (3,6) and (6,3) represent the same combination of two dice rolls. It's not that you can't tell which is which, but the fact that it simply doesn't matter.

This question might be a little more difficult: how many combinations are there, in total, when you roll these two dice? ***21. Each of the six "doubles" represents their own combination. Draw a diagonal line connecting these six cells. Then, for every cell below this line, there is a corresponding cell above the line that, together, creates one combination. So, in order to get the number of combinations, one only needs to count those on the line and either below or above the line. There are 1+2+3+4+5 below the line, and then the 6 on the diagonal. So the sum of those numbers is 21.***

That seems like an odd answer, 21. How could we check our answer? Well, we know the probability of every combination, so if we have them all, their sum should equal 1! We know the probability of each set of doubles is 1/36. So we have 6*(1/36) = 1/6. Each of the other 15 combinations has a probability of 2/36 = 1/18, right? So we have 15*(1/18) = 15/18 = 5/6. 1/6 + 5/6 = 1! So that's all of them.

Let's up the difficulty a little bit. Say that instead of rolling these two dice, we rolled two dice with 20 faces on each. These do exist, unfortunately. So how many permutations are there for the rolling of these two dice? Try to answer without counting them individually. That could take you a while. ***There are 400 permutations. We could create a similar table as before, except now it's 20x20. 20*20=400.***

Alright then. What about combinations for these two dice? It may help you to draw the skeleton of the diagram similar to the one we drew before, but again I urge you not to fill in every single permutation. See if you can find a pattern. ***There are 210 combinations. There are 20 ways to get doubles along the diagonal of the table. Then below the line there are 1, then 2, then …, then 19 combinations. So the total number is 1 + 2 + … + 19 + 20. This is equal to 210.***

Very well done. One more set of questions, and then I promise we're done. What if these dice had *n* sides? How many permutations would there be rolling two of these dice? You'll need to have a rule for this; the answer obviously won't be a number. ***There will be n^2 permutations.***

That seemed pretty straightforward. What about the number of combinations? You don't need to come up with the most elegant formula; just try to come up with some way of representing it, then we can work together to come up with the prettiest answer. ***There will be 1 + 2 +…+(n-1) + n combinations.***

Yes, that's correct. Now, what is that equal to? Can we make that look a little better? Let's set that quantity equal to S, for "sum":

$$S = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

Now, let's try to find another way to represent S. We know that addition is commutative, right? We know 1 + 2 + 3 is the same as 2 + 3 +1 is the same as 3 + 1 + 2. How can we rearrange S that might help us find out what it is? What is one of the simplest things we could do? ***Write everything backwards?***

Let's do that! Now we have:

$$S = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

$$S = n + (n-1) + (n-2) + \dots + 3 + 2 + 1$$

Do you notice anything? ***Adding down gives (n+1) for everything.***

Yes! Let's just add the first and second rows. This gives us:

$$2S = (n+1) + (n+1) + (n+1) + \dots +(n+1) + (n+1) + (n+1)$$

How many (n+1)'s are there? ***There are n terms in S, so there are n (n+1)s.***

So how can we simplify this? ***2S = n(n+1).***

So what do we get for S? ***S = (n(n+1))/2.***

Wait, what does that mean, again? ***S is the number of combinations when rolling two n-sided dice.***

So this is great! Now we know the number of permutations and combinations when rolling two dice that have the same number of sides, as long as they're the same. So which is greater? ***Permutations > Combinations. n^2 > n^2/2 + n/2.***